# Transient Time in Unidirectional Synchronisation

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**Abstract**

We consider here the behaviour of a dynamical system consisting of unidirectionally coupled Duffing oscillators. Under certain conditions the subsystems may become synchronised corresponding to a stable invariant subset of the full dimensional phase space. The distribution for the transient time that trajectories take to converge onto the synchronised state is investigated via numerical simulations. Some initial conditions undergo a very long transient motion distinct from the drive, before synchronisation occurrs. The dependence of this time on the governing system parameters and on the initial conditions of the driving system is discussed.

Short title: Transient Time to synchronisation.

# 1 Introduction

When considering a system of nonlinear equations, the dynamics may show convergence towards a stable invariant subset if the Lyapunov spectrum transverse to that subset is negative [Pecora & Carroll 1990, Heagy *et al.* 1994b]. The spatial evolution and transient phenomena before a trajectory may settle onto the subset is largely unknown, especially since complex phenomena such as *intermittency* [Heagy *et al.* 1994a, Ott & Sommerer 1994] and *riddled basins* [Alexander *et al.* 1992, Sommerer & Ott 1993] may occur. In this paper we report some detailed simulations which focus on the convergence of the dynamics onto the invariant subset, looking at possible relationships between exponents of scaling laws and the relative change of parameters and/or initial conditions.

To illustrate some observed characteristics we investigate a system consisting of two separate Duffing oscillators with identical parameters which are coupled through a unidirectional linear relationship. Our interest is focused on the behaviour of the system when it is in a chaotic state with values of the coupling parameter which lead to synchronisation of the two oscillators and, in particular, in the time of transient decay onto the synchronised state. We shall discuss a simple problem: if we have a system with a stable synchronised state, how long does a trajectory take to converge onto it? Different initial conditions may converge at different rates, thus it may be important to assess not only the rate of convergence but also any spatial variation. It is also important to examine how these two aspects alter as we vary system parameters. A particular system but viewed at different parameters, or alternative systems, may also have different degrees of stability for the invariant subset, influencing the convergence of the trajectory towards the synchronised state. Two different notions of stability have been recently pointed out for such invariant subsets, called states of *asymptotic stability* and *monotonic stability* [Kapitaniak & Thylwe 1996]. Asymptotic stability is a condition achieved when the chaotic attractor in the invariant subset is Lyapunov stable and its basin of attraction (for the transverse stability) contains a neighbourhood of the attractor, while in terms of the transverse Lyapunov spectrum, asymptotic stability is a state for which the supremum of all transverse exponents, computed for all different ergodic probability measures supported in the attractor, is negative [Ashwin *et al.* 1996]. On the other hand, monotonic stability is given in terms of the scaling of the distance (not necessarly the Euclidean one) between the trajectories of the subsystems, *i.e.* the stability of the chaotic attractor in the invariant subset is *monotonic* if the distance goes to zero monotonically decreasing in time [Kapitaniak & Thylwe 1996].

Our study will discuss if there is a particular distribution of the transient time to synchronisation when the degree of stability varies (*i.e.* as we change a coupling parameter, the maximum transverse Lyapunov exponent may become smaller in magnitude, weakening the stability) and, for fixed parameters, in the space of initial conditions of the driving system. In Gupte & Amritkar [1993] the authors stated that *"The length of the transient after which the system settles down onto the desired orbit depends on the value of the largest SLE (Sub-Lyapunov Exponent, sic.) of the response system"*, pointing out that, as in the case of much simpler attracting solutions, the speed of convergence towards a stable invariant subset depends on the contraction rate in the direction transverse to the subset.

## 2 Coupled Duffing Equations

A single driven Duffing oscillator,

$$\ddot{x} + \mu\dot{x} - x + x^3 = A\cos(\omega t) \tag{1}$$

is a second-order non autonomous differential equation that describes the motion of a unit mass particle in a double-well potential, subject to viscous damping and forced by a cosinusoidal term [Thompson & Stewart 1986]. For the purposes of this paper, the parameters are fixed at $\mu = 0.1$, $\omega = 0.3$, and $A = 3.0$ producing a cross-well chaotic response, which is the only stable attractor. Two identical oscillators are then coupled together, with a unidirectional coupling type to yield:

$$\begin{cases} \ddot{x} + \mu\dot{x} - x + x^3 = A\cos(\omega t) \\ \ddot{y} + \mu\dot{y} - y + y^3 + K(y - x) = A\cos(\omega t). \end{cases} \tag{2}$$

The Eqs. (2) represent an extended system in which, in addition to the forcing term $A\cos(\omega t)$, the $x$ system can be thought of as a driving for the slave or response $y$ system through the coupling term, where $K$ is a real non-negative parameter. This type of coupling produces an extended 5-dimensional system for which the invariant set $x(t) = y(t)$ exists, *i.e.* a synchronised motion.

To illustrate the synchronisation property (for the moment regardless of the time to convergence) we consider the parameters for both systems to be identical as previously defined. Thus, without loss of generality, if we select initial conditions for the driving system ($x$) to be $(x_0, \dot{x}_0) = (1.1, -0.5)$ the variable $x$ will undergo a chaotic motion. The initial conditions for the slave system ($y$) are set at $(y_0, \dot{y}_0) = (1.0, -0.9)$, and using a usual Runge-Kutta numerical scheme the full system (2) may be integrated. The Euclidean distance $d = \sqrt{(x-y)^2 + (\dot{x} - \dot{y})^2}$ between the two trajectories is monitored for various choices of the coupling parameter $K$, as shown in Figure 1. In this plot the system is set to run for approximately 70 cycles of the periodic forcing to allow for the decay of transients. Over the next 10 cycles an average of the mean value between successive maxima and minima of the function $d(t)$ is evaluated. For $K = 0$ the two systems are independent and they show an average distance $\overline{d} \simeq 1.5$ (approximatively half of the size of the attractor); increasing the value of the coupling parameter we see from the average distance that the transition to the synchronised state occurs at $K_{thr} \simeq 2.0$ for this numerical experiment, whereafter the two subsystems display the same output. For values of $K$ greater than $K_{thr}$ the synchronised state is stable. This scenario is specifically for the initial conditions given but different initial conditions would qualitatively produce the same response.

Defining a precise threshold value (*i.e.* for all initial conditions) of the coupling parameter for the convergence of the dynamics onto an invariant subspace involves two intrinsic problems:

1. For low values of $K$ the invariant set is strongly unstable and for high values is strongly stable, while for values of the coupling constant close to the threshold, weak stability/instability causes the basin of attraction for the convergence onto the subset to have a very complicated structure, sometimes *riddled*, even in regions very close to the invariant set. So distinct pairs of initial conditions $(x_0, \dot{x}_0)$, $(y_0, \dot{y}_0)$ may lead to slightly different values for $K_{thr}$.

2. Simulations for locating the synchronisation threshold, or for computing characteristic exponents, are produced by running the integration over a finite time, supposed to be large enough to avoid problems of transient time. The length of the transient time may depend on the coupling parameter and on the initial conditions, and as yet no distribution is known for this latter.

Returning to the question of the time to synchronisation, in relation to the second problem above, we plot, in Figure 2, two typical curves representing the full evolution of $d(t)$ (in linear-log scale), for two values of $K$ above the synchronisation threshold, now without ignoring the first 70 cycles. The time, here expressed in cycles of the periodic forcing ($2\pi/\omega$), has been left running until $d(t) = 0$ exactly up to the precision of the computer. The two curves are computed for identical initial conditions (given in the caption) but $K = 2.7$ and $K = 2.05$, so that the invariant subset has two different degrees of stability, and we note that the observed decay is qualitatively different. The evolution of $d(t)$ when $K = 2.7$ can be notionally split into three different parts. The first evolution ($\tau_o$), until approximatively 50 cycles of the periodic forcing, shows no appreciable change in the order of magnitude of the distance measure $d(t)$ (about unity), and in fact it is related to the chaotic wandering that the response system is performing about the attractor of the driving system, so that an approximate horizontal line can be seen. After this, in the time interval $\tau_d$, the trajectory starts to decay towards the synchronised state, and in logaritmic scale its decay is almost linear, demonstrating an exponential dependence of the form $d(t) \propto \exp(\lambda t)$, with a rate of contraction $\lambda < 0$. After approximatively 80 cycles $d(t)$ reaches the level of roundoff error, where a pseudo-random oscillatory phenomenon takes place for values of $d(t)$ around $10^{-15}$. We refer to the first non-decaying part of the transients $\tau_o$ as the *orbiting transient*, while the second part $\tau_d$ the *decaying transient*. When $K = 2.05$ the degree of stability is weaker than in the previous case (this statement can be quantified by the largest transverse Lyapunov exponent), and this condition is reflected in the decay of $d(t)$, for which a division of the trajectory into qualitatively different decaying parts is no longer possible. We consider the system in the first case ($K = 2.7$) to be in a state of monotonic stability, while for the second case ($K = 2.05$) the system shows asymptotic stability. The condition of monotonic stability, as reported in Kapitaniak & Thylwe [1996], is that $d(t)$ must be a monotonically decreasing function of time; a very strict condition that we are actually taking, in a more liberal sense, to include the orbiting transient and the overall oscillatory motion even during the decaying part of the transient. It seems reasonable that for the monotonic stability the decaying part of the transient is an exponential function of time, with the maximum transverse Lyapunov exponent as coefficient of the exponential. We show in the following that a linear fit of the decaying transient in logaritmic scale is very close to the maximum transverse exponent. What is actually less clear is if there is any distribution for the orbiting transient. Another intuitive comment is that when the motion in the invariant subset posesses a weaker stability, transients will produce longer relaxation times, so we might expect longer orbiting transients on average, a condition that is already fulfilled by the decaying part of the transient.

The orbiting transient is dependent on the intial conditions chosen. In Figure 3 we show curves representing the evolution of $d(t)$, again on linear-log scale, with $K = 2.7$ fixed, for three different initial conditions. More precisely, $(y_0, \dot{y}_0) = (1.1, -0.5)$ and $\dot{x}_0 = -1.1$ are kept fixed, and we use three different starting points $x_0$, as given in the figure below each end point of the three curves. The slope of the linearly decaying part or the decaying transient for each of the three curves is almost the same, corresponding to the intuitive conjecture that the convergence is governed by the strength of dissipation transverse to the attractor. However, the three trajectories shown converge in very different times; almost a delay of approximatively 50 cycles of the forcing for $x_0 = 3.1$ when compared to the trajectory starting at $x_0 = 1.1$. If we look closer at the trajectory of the response system in the phase space, we see that this is not just simply orbiting about the attractor of the driving, but rather it is performing a different orbit, larger in size that the orbit of the driving system. A transient trajectory of the response system $(y, \dot{y})$ before synchronisation is achieved is shown in Figure 5. This tyoe of trajectory (either steady state or transient) has not been seen in the single Duffing oscillator. We conjecture that this orbiting transient is following a path close to either of an unstable orbit of the extended 5-dimensional system, or perhaps the path of a solution (stable or unstable) that no longer exists, but which can be found in some nearby region of parameter space.

# 3 Transient Distribution

We have seen in Figure 2 that not all transients behave the same but, for all cases in which we have been able to divide the trajectory into different parts (*e.g.* the curve for $K = 2.7$ in Figure 2, and all curves in Figure 3), we can linearly fit the decaying part ($\tau_d$) in linear-log scale. The comparison between the slope of this fit and the maximum transverse Lyapunov exponent (denoted TLE) is given in Figure 4 for values of $K$ between 2.3 and 3.0 in steps of 0.1. For $K < 2.3$ a trajectory may still undergo orbiting and decaying transient behaviour, as described in the previous section, but the evaluation of a suitable linear fit becomes much less accurate. Moreover, progressively lowering the coupling parameter, as we approach $K_{thr}$ from above, we expect to find decaying trajectories which resemble the curve for $K = 2.05$ in Figure 2. The maximum transverse Lyapunov exponent $\lambda_{\perp}$ has been computed as rate of contraction of perturbations transverse to the synchronised state, *i.e.* as the Lyapunov exponent of the difference system $(x - y, \dot{x} - \dot{y})$ [Ashwin *et al.* 1996]. The values reported for the fit $\lambda$, and for the maximum transverse Lyapunov exponent $\lambda_{\perp}$ have been obtained as an average of 20 different initial conditions for each value of $K$. All the values obtained for the fit were very close to the average, with a standard deviation of the order of $10^{-4}$ for almost all values of $K$. The estimate for $K = 2.3$ showed a less precise fit, with standard deviation $5 \times 10^{-4}$.

Our aim is to determine whether, once all parameters are fixed, there is any particular spatial or time distribution for the transient time given a representative set of initial conditions in the phase space. Again using the fixed parameters given in Eqs. (2), and fixed $K$, we set the initial conditions for the response system to be $(y_0, \dot{y}_0) = (1.1, -0.5)$ and we vary $(x_0, \dot{x}_0)$ denoting $x_0 = y_0 + \xi_0$ and $\dot{x}_0 = \dot{y}_0 + \eta_0$. We follow here the natural choice of varying the initial conditions of the driving system. Setting a grid of $(\xi_0, \eta_0)$ values, for each $(\xi_0, \eta_0)$ chosen, we run time forward until the Euclidean distance $d(t)$ reaches a cutoff value of $10^{-6}$. For $K$ fixed the slope of the decaying transient, in linear-log scale, is almost the same for all initial conditions, so a fixed cutoff will give us a faithful representation of the orbiting transient times $\tau_o$.

The results of the computations carried out for the case $K = 3.0$ are given in Figure 5. On the upper left diagram we show a histogram describing how all initial conditions in the grid distribute themselves with respect to their orbiting time $\tau_o$. This "density" plot shows a peak at about $\tau_o \simeq 8$, and we can ideally divide the distribution into three parts; as a distribution before the peak (these are the points whose synchronisation is the fastest), then the peak, that contains almost all the "mass" of the distribution, and then finally the tail, formed by all points achieving synchronisation in the longest time possible (at least, within the grid we set). For $K = 3.0$, initial conditions which achieve synchronisation slowest take about 40 cycles of the periodic forcing. The picture in the top right of Figure 5 shows a spatial organization of the initial conditions whose synchronisation is in the pre-peak region, arbitrarly taken at $\tau_o < 6$ cycles of the forcing. The initial conditions are taken in the range $[\xi_0, \eta_0] = [-6, 6] \times [-8, 8]$. The first thing we can notice is that the distribution of points shows some organized shape, indicating the existence of privileged zones of the phase space in which the convergence towards the invariant subset is the fastest, taking into account also the arbitrarity of the section cut. To check the robustness of the distribution in the presence of noise, we have repeated the same procedure as above but introducing noise in both the drive and response systems. The amplitude of the noise has been increased up to $10^{-5}$, without any considerable change in the shape of the pre-peak section, and in some enlargments of it.

The two lower pictures represent the dynamics of the drive (left) and response (right) systems in the phase space for a trajectory which displays very delayed synchronisation. The two pictures are shown to the same scale, to illustrate the difference of the pre-synchronised transient. The left picture shows a trajectory of the cross-well motion typical of the Duffing oscillator, for initial conditions of the driving $(x_0, \dot{x}_0) = (1.6, -1.1)$. The first 50 cycles of the response system are shown in the lower right part of Figure 5, for initial conditions $(y_0, \dot{y}_0) = (1.1, -0.5)$. This picture is produced using the same initial points as the third time history in Figure 3, where a $\tau_o$ of approximatively 80 cycles of the forcing are displayed. In this case the value of $K$ is different, but still the numerical integration produced a time history with very long orbiting transient. For $t < \tau_o$ the orbit displays the spatial evolution of the orbiting transient before convergence to synchronisation state, after which the trajectory rapidly settles onto the attractor displayed in the lower left picture. The orbit of the response is not a simple chaotic wandering about the attractor of the driving, but instead it performs a different orbit, before suddenly converging onto the driving.

As far as we can deduce, this orbiting transient is not following a stable orbit of the single Duffing equation. The fact that noise does not strongly influence the overall behaviour seems to indicate that neither it is following an unstable orbit of the full 5-dimensional system. This leaves the possibility that an orbit exists in nearby parameter space onto which the transient orbit becomes trapped for a lenght of time before synchronisation occurrs.

# 4 Conclusions

In this paper we have considered the time to synchronisation of two unidirectionally coupled chaotic systems, using the driven Duffing oscillator as a demonstrative tool. Specifically, the effect of changes in the coupling parameter and initial conditions have been investigated. For this particular system for values of the coupling parameter greater than the threshold for stability of the invariant set corresponding to synchronised motion, trajectories undergo a transient motion which can be split in two parts. Once close to the invariant manifold, trajectories decay at a rate governed by the TLE. However, prior to this decay, a transient may "orbit" the invariant manifold for an almost indeterminate time. It is conjectured that an orbit may exist in the nearby parameter space of the full 5D system which is not a solution of the 3D driving system, giving rise to transients which follow this orbit before decaying onto the steady state attractor, so the convergence onto the synchronised state would not only be governed by the dynamics within the invariant subset, but also depends upon the global dynamics of the full phase space.

Here we have investigated the spatial distribution in the initial condition space of trajectories which converge to the synchronised state in specific times. Further work needs to be carried out to identify the full underlying dynamic which determinate global behaviour.


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Figure 1: Plot of several values of the Euclidean distance $d(t)$ between the trajectories $(x(t), \dot{x}(t))$ and $(y(t), \dot{y}(t))$ for different values of $K$. The transition to a stable synchronised state is located approximatively at $K_{thr} = 2.0$.

Figure 2: Two curves representing the time evolution of the Euclidean distance $d(t)$ between the drive and the response trajectories for the parameter values $\mu = 0.1$, $\omega = 0.3$, $A = 3.0$ with coupling parameter $K = 2.05$ and $K = 2.7$. The initial conditions that generate these two plots are the same, namely $(x_0, \dot{x}_0) = (3.1, -1.1)$ and $(y_0, \dot{y}_0) = (1.1, -0.5)$, but the different value of the coupling constant leads to qualitatively different decay onto the invariant subset. For the curve with $K = 2.7$ we have labelled the time intervals of the orbiting transient as $\tau_o$ and decaying transient as $\tau_d$.

Figure 3: The behaviour of $d(t)$ for three cases of convergence onto the synchronised subset of Eqs. (2) for the parameter values as in the previous Figures, $A = 3.0$ and $K = 2.7$. In all cases the initial conditions for the slave system are set to $(y_0, \dot{y}_0) = (1.1, -0.5)$. For the driving system we set $\dot{x}_0 = -1.1$ and choose three different values for $x_0$ as reported below each plot.

Figure 4: Estimates of the linear fit compared to the transverse Lyapunov exponent (TLE) for a variation of $K$. The circles represent the computed value of the maximum, while the squares represent the fitted values of the slope of the decaying transient for a trajectory in linear-log scale as in Figure 3. The linear fit is averaged over 20 initial conditions for each value of $K$, and the respective error bars are shown.

Figure 5: Upper panel: Time distribution of the initial points $(\xi_0, \eta_0)$ for increasing orbiting time $\tau_o$ for $K = 3.0$, and spatial organization of the initial conditions in the pre-peak region of the previous histogram, in the $(\xi_0, \eta_0)$ plane. Lower panel: Attractor of the drive system (left) and a phase plane representation of the trajectory of the response system (right) in its state of orbiting transient before synchronisation takes place.